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Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study

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The article develops the notion of covariational reasoning and proposes a framework for describing the mental actions involved in applying covariational reasoning when interpreting and representing dynamic function events. It also reports on an investigation of high-performing 2nd-semester calculus students' ability to reason about covarying quantities in dynamic situations. The study revealed that these students were able to construct images of a function's dependent variable changing in tandem with the imagined change of the independent variable, and in some situations, were able to construct images of rate of change for contiguous intervals of a function's domain. However, students appeared to have difficulty forming images of continuously changing rate and could not accurately represent or interpret inflection points or increasing and decreasing rate for dynamic function situations. These findings suggest that curriculum and instruction should place increased emphasis on moving students from a coordinated image of two variables changing in tandem to a coordinated image of the instantaneous rate of change with continuous changes in the independent variable for dynamic function situations.

Key Words: Applications; Calculus; Cognitive development; College mathematics; Functions; Mathematical modeling; Reasoning

Since the late19th century, there have been repeated calls to increase the emphasis on functions in school curricula (College Entrance Examination Board, 1959; Hamley, 1934; Klein, 1883). More recently, the literature on early function instruction supports the promotion of conceptual thinking about functions that includes investigations of patterns of change (Kaput, 1994; Monk, 1992; NCTM, 1989, 2000; Sfard, 1992; Thorpe, 1989; Vinner & Dreyfus, 1989). Both in 1989 and in 2000, the authors of the National Council of Teachers of Mathematics (NCTM) *Standards* documents called for students to be able to analyze patterns of change in various

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contexts. They recommended that students learn to interpret statements such as "the rate of inflation is decreasing," and in general promoted the idea that students should develop a "deeper understanding of the ways in which changes in quantities can be represented mathematically" (NCTM, 2000, p. 305). In addition, the authors of the *National Science Education Standards* (National Research Council, 1996) have called for students to use mathematical functions to identify patterns and anomalies in data (p. 174).

It is not clear to what extent mathematics curricula have responded to these calls (Cooney & Wilson, 1993). Research suggests that undergraduate students are entering the university with weak understandings of functions, and entry-level university courses do little to address this deficiency (Carlson, 1998; Monk, 1992; Monk & Nemirovsky, 1994; Thompson, 1994a). Recent investigations of college students' understandings of functions have documented that even academically talented undergraduate students have difficulty modeling functional relationships of situations involving the rate of change of one variable as it continuously varies in a dependent relationship with another variable (Carlson, 1998; Monk & Nemirovsky, 1994; Thompson, 1994a). Research has also shown that this ability is essential for interpreting models of dynamic events (Kaput, 1994; Rasmussen, 2000) and is foundational for understanding major concepts of calculus (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Kaput, 1994; Thompson, 1994a; Zandieh, 2000) and differential equations (Rasmussen, 2000).

In studying the process of acquiring an understanding of dynamic functional relationships, Thompson (1994b) has suggested that the concept of rate is foundational. According to Thompson, a mature image of rate involves the following: the construction of an image of change in some quantity, the coordination of images of two quantities, and the formation of an image of the simultaneous covariation of two quantities. These phases parallel Piaget's three-stage theory about children's mental operations involved in functional thinking about variation (Piaget, Grize, Szeminska, & Bang, 1977). Also contributing to our understanding of the notion of covariation is the work of Saldanha and Thompson (1998), who describe understanding covariation as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (p. 298). This mental activity involves the coordination of the two quantities, then tracking either quantity's value with the realization that the other quantity also has a value at every moment in time. In this theory, images of covariation are viewed as developmental, with the development evolving from the coordination of two quantities to images of the continuous coordination of both quantities for some duration of time. According to Saldanha and Thompson (1998), "In early development, one coordinates two quantities' values—think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities values persist" (p. 298).

Confrey and Smith (1995) see a covariation approach to creating and conceptualizing functions as involving the formation of links between values in a function's domain and range. In the case of tables, it involves the coordination of the variation in two or more columns as one moves up and down the table (Confrey & Smith, 1994). For both Confrey and Smith (1995) and Thompson (1994a), *coordinating* is described as foundational for reasoning about dynamic function relationships. Even though the covariation of two quantities does not always require the notion of time, it is through the metaphor of "exact time" for the imagined location of a "moving point" that Confrey and Smith and many others discuss covarying quantities (e.g., Monk's [1992] "across-time" function view and Nemirovsky's [1996] variational versus pointwise approach).

In this article, we propose a framework for the study of covariational reasoning and illustrate how this framework can be used to analyze students' understanding about dynamic situations involving two simultaneously changing quantities. We also present problems that evoke and require the use of covariational reasoning, and in doing so, we illustrate features of curricula that emphasize a covariational approach to learning functions. We describe our research findings about high-performing 2nd-semester calculus students' covariational reasoning abilities and discuss implications of these results.

DEFINITIONS

On the basis of these studies and our research from the last several years (Carlson, 1998; Carlson, Jacobs, & Larsen, 2001; Carlson & Larsen, in press), we define *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other. We concur with Saldanha and Thompson's (1998) view that images of covariation are developmental, and we use the term *developmental* in the Piagetian sense (Piaget, 1970) to mean that the images of covariation can be defined by level and that the levels emerge in an ordered succession. Throughout this paper, our use of the word *image* is consistent with the description provided by Thompson (1994b). The construct of image is portrayed as "dynamic, originating in bodily actions and movements of attention, and as the source and carrier of mental operations" (p. 231). The notion of image is not inconsistent with that of *concept image* as defined by Vinner and Dreyfus (1989; i.e., the mental pictures, visual representations, experiences, properties, and impressions associated with a concept name by an individual in a given context); however, its focus is on the dynamics of mental operations (Thompson, 1994b). We use the word *rate* to mean the average rate of change in the case of imagining a subinterval, or the instantaneous rate of change in the case of imagining a function over its entire domain.

The terms *pseudo-analytical thought processes* and *pseudo-analytical behaviors* identify, respectively, processes of thought and behaviors that take place without understanding, and pseudo-analytic behaviors are produced by pseudo-analytical thought processes (Vinner, 1997). According to Vinner (1997), "Pseudo-analytic behaviors describe a behavior which might look like conceptual behavior, but which in fact is produced by mental processes which do not characterize conceptual behavior" (p. 100); these behaviors and thought processes are not necessarily negative and may be the result of "spontaneous, natural, but uncontrolled associations" (p. 125). Pseudo-analytical behaviors differ from *pseudo-conceptual* behaviors in that the focus of the former is on the analytic process rather than the concept; however, these two ideas should not be seen as mutually exclusive as there are some contexts in which both the analytical and conceptual modes are involved.

BACKGROUND

In recent years, our understanding of ways in which college students interpret and represent dynamic function situations has been informed by considerable research (Carlson, 1998; Kaput, 1994; Monk, 1992; Nemirovsky, 1996; Sierpinska, 1992; Thompson, 1994b). In examining the thinking of calculus students who are attempting to interpret the changing nature of *rate of change* for intervals of a function's domain, several studies (Carlson, 1998; Monk, 1992; Monk & Nemirovsky, 1994; Nemirovsky, 1996; Thompson, 1994a) have revealed that this ability is slow to develop, with specific problems reported in students' ability to interpret graphical function information. Studies by Monk (1992) and Kaput (1992) have noted that calculus students show a strong tendency to become distracted by the changing shape of a graph and in general do not appear to view a graph of a function as a means of defining a covarying relationship between two variables. Other studies have found that calculus students have difficulty interpreting and representing concavity and inflection points on a graph (Carlson, 1998; Monk, 1992). Even when directly probed to describe their meaning in the context of a dynamic real-world situation, students made statements such as "second derivative positive, concave up," and "second derivative equal to zero, inflection point" (Carlson, 1998). Further probing revealed that these students appeared to have no understanding of why this procedure worked and in general did not appear to engage in behaviors that suggested that they were coordinating images of two variables changing concurrently. Tall (1992) also found that, although college students' concept images of function included a correspondence notion, the idea of operation, an equation, a formula, and a graph, it did *not* include the conception of two variables changing in tandem with each other.

Research into students' developing conceptions of function has revealed that a view of function as a process that accepts input and produces output (Breidenbach, Dubinsky, Hawks, & Nichols, 1992) is essential for the development of a mature image of function. This view has also been shown to be foundational for coordinating images of two variables changing in tandem with each other (Carlson, 1998; Thompson, 1994a). According to Thompson (1994a),

> Once students are adept at imagining expressions being evaluated continually as they "run rapidly" over a continuum, the groundwork has been laid for them to reflect on a set of possible inputs in relation to the set of corresponding outputs. (p. 27)

The covariation view of function has also been found to be essential for understanding concepts of calculus (Cottrill et al., 1996; Kaput, 1992; Thompson, 1994b; Zandieh, 2000). Students' difficulties in learning the limit concept have been linked to impoverished covariational reasoning abilities. In a recent study, Cottrill et al. (1996) recommended that the limit concept should begin with the informal dynamic notion of the "values of a function approaching a limiting value as the values in the domain approach some quantity" (p. 6). The development of this "coordinated" process schema of limit was found to be nontrivial for students, and their difficulty in grasping it has been sited as a major obstacle to their developing conception of limit.

In describing her framework for analyzing students' understanding of derivative, Zandieh (2000) also suggested that a view of function as the covariation of the input values with the output values is essential. In her framework, she stated that "the derivative function acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at that point" (p. 107), emphasizing the notion that the derivative function results from covarying the input values of the derivative function with the rate-of-change values of the original function.

Thompson (1994b) suggested that covariational reasoning is foundational for students' understanding of the Fundamental Theorem of Calculus: "The Fundamental Theorem of Calculus—the realization that the accumulation of a quantity and the rate of change of its accumulation are tightly related is one of the intellectual hallmarks in the development of the calculus" (p. 130). When interpreting the information conveyed by a speed function, the total distance traveled relative to the amount of time passed is imagined as the coordination of accruals of distance and accruals of time.

Collectively, these studies suggest that covariational reasoning is foundational for understanding major concepts of calculus and that conventional curricula have not been effective in promoting this reasoning ability in students. Building on these findings, our study investigated the complexity of constructing mental processes involving the rate of change as it continuously changes in a functional relationship. A framework for investigating covariational reasoning is described in the next section.

THEORETICAL FRAMEWORK

Covariational Reasoning

A description of the five mental actions of covariational reasoning and the associated behaviors are provided in Table 1. The listed behaviors have previously been identified in undergraduate students while they were responding to tasks that involve interpreting and representing dynamic function situations (Carlson, 1998).

The mental actions of the covariation framework provide a means of classifying behaviors that are exhibited as students engage in covariation tasks; however, an individual's covariational reasoning ability relative to a particular task can be determined only by examining the collection of behaviors and mental actions that

Table 1
Mental Actions of the Covariation Framework

Mental action	**Description of mental action**	**Behaviors**
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• Constructing an increasing straight line • Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• Plotting points/constructing secant lines • Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• Constructing contiguous secant lines for the domain • Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	• Constructing a smooth curve with clear indications of concavity changes • Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

were exhibited while responding to that task. A student is given a level classification according to the overall image that appears to support the various mental actions that he or she exhibited in the context of a problem or task. The Covariation Framework contains five distinct developmental levels (Table 2). We say that one's covariational reasoning ability has reached a given level of development when it supports the mental actions associated with that level *and* the actions associated with all lower levels.

The notion of image used in describing the levels of the framework is consistent with Thompson's (1994a) characterization of an image as that which "focuses on the dynamics of mental operations" (p. 231). As an individual's image of covariation develops, it supports more sophisticated covariational reasoning. (Recall that we define *covariational reasoning* to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other).

A student who is classified as exhibiting Level 5 (L5; i.e., Instantaneous Rate Level) covariational reasoning, relative to a specific task, is able to reason using

Table 2
Levels of the Covariation Framework

Covariational Reasoning Levels

The covariation framework describes five levels of development of images of covariation. These images of covariation are presented in terms of the mental actions supported by each image.

Level 1 (L1). ***Coordination***

At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).

Level 2 (L2). ***Direction***

At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are *both* supported by L2 images.

Level 3 (L3). *Quantitative Coordination*

At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions identified as MA1, MA2 and MA3 are supported by L3 images.

Level 4 (L4). *Average Rate*

At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions identified as MA1 through MA4 are supported by L4 images.

Level 5 (L5). *Instantaneous Rate*

At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change. It also includes awareness that the inflection point is where the rate of change changes from increasing to decreasing, or decreasing to increasing. The mental actions identified as MA1 through MA5 are supported by L5 images.

MA5 and is also able to unpack that mental action to reason in terms of MA1 through MA4. He or she is able to coordinate images of the continuously changing rate with images of continuous changes in the independent variable and is able to describe the changing nature of a dynamic event in terms of MA3 and MA4 (Note: MA3 includes MA1 and MA2). This image of covariation (i.e., L5 reasoning) supports behaviors that demonstrate that the student understands that the instantaneous rate of change resulted from smaller and smaller refinements of the average rate of change and that an inflection point is where the rate of change changes from increasing to decreasing, or from decreasing to increasing.

We note that some students have been observed exhibiting behaviors that gave an appearance of engaging in a specific mental action; however, when their behaviors were probed, these students did not provide evidence that they possessed an understanding that supported the behavior. We refer to such behavior as pseudo-analytical behavior (i.e., the underlying understanding necessary for performing the specific behavior meaningfully is not present [Vinner, 1997]), and we describe the mental action that produced the behavior as a pseudo-analytical mental action. (Recall that, according to Vinner [1997], pseudo-analytic behavior is produced by pseudo-analytical thought processes). We also reemphasize that a student is classified as having a specific covariational reasoning ability level (say, L5) only if she or he is able to perform the mental action associated with that level (MA5) and all lower-numbered mental actions (MA1 through MA4). In other words, it is possible for a student to exhibit MA5 without applying L5 covariational reasoning, and an example of this will be described later in the article.

The proposed covariation framework provides an analytical tool with which to evaluate covariational thinking to a finer degree than has been done in the past. In addition, it provides a structure and language for classifying covariational thinking in the context of a student's response to a specific problem, and for describing a student's general covariational reasoning abilities (i.e., developmental level in the framework).

Covariational Reasoning in a Graphical Context

Students' covariational reasoning abilities are important for interpreting and representing graphical function information. Because these activities related to graphs have been the context in which we initially observed student difficulties, they have been the focus of much of our work context. A close look at students' covariational reasoning in the context of a graph reveals that students who exhibit behaviors supported by MA1 typically recognize that the value of the *y*-coordinate changes with changes in the value of the *x*-coordinate. Typically, the *x*-coordinate plays the role of the independent variable, although we have observed students treating the *y*-coordinate as the independent variable. This initial coordination of the variables is commonly revealed by a student labeling the coordinate axes of the graph, followed by utterances that demonstrate recognition that as one variable changes the other variable changes. Attention to the direction of change (in the case of an increasing function) involves the formation of an image of the *y*-values getting higher as the graph moves from left to right (MA2, Table 1). In our experience, the common behavior displayed by students at this level has been the construction of a line that rises as one moves to the right on the graph or utterances that suggest an understanding of the direction of change of the output variable while considering increases in the input variable (e.g., as more water is added, the height goes up). MA3 involves the coordination of the relative magnitudes of change in the *x* and *y* variables. In this context, students have been observed partitioning the *x*-axis into intervals of fixed lengths (e.g., x_1, x_2, x_3, x_4) while considering the amount of change in the output for each new interval of the input. This behavior has been commonly followed by the student's construction of points on the graph (the student views the points as representing amounts of change of the output while considering equal amounts of the input), and this behavior is followed by his or her construction of lines to connect these points. Activity at the rate level involves recognition that the amount of change of the output variable with respect to a uniform increment of the input variable expresses the rate of change of the function for an interval of the function's domain. This recognition is typically revealed by the student's sketching of secant lines on a graph or by carrying out the mental computation or estimation of the slope of a graph over small intervals of the domain (the sketching of these lines would result from the student imagining and adjusting slopes for different intervals of the domain). It is noteworthy that mental actions identified as MA3 and MA4 may both result in the construction of secant lines; however, the type of reasoning that produces these constructions is different (i.e., MA3 focuses on the *amount* of change of the output (height) while considering changes in the input; and MA4 focuses on the *rate* of change of the output with respect to the input for uniform increments of the input). Attention to continuously changing *instantaneous rate* (MA5) is revealed by the construction of an accurate curve and includes an understanding of the changing nature of the instantaneous rate of change for the entire domain. It should be noted that a student may perform MA5 without demonstrating an understanding that the instantaneous rate of change resulted from examining smaller and smaller intervals of the domain. However, the developmental nature of the framework indicates that only students who are able to unpack MA5 (build from MA1 to MA4) would receive a L5 covariational reasoning classification. This L5 image has been shown to support an understanding of *why* a concave-up graph conveys where the rate of change is increasing and *why* the inflection point relates to the point on the graph where the rate of change changes from increasing to decreasing, or from decreasing to increasing.

Use of the Framework

This section provides information based on a dynamic situation shown in Figure 1 and called the Bottle Problem, which illustrates common covariational reasoning behaviors that have been expressed by students when responding to a specific task (Carlson, 1998; Carlson & Larsen, in press). The mental actions supported by each image of covariation are followed by a description of specific behaviors that have been observed in students and their corresponding classifications in using the framework.

The *Coordination Level* (L1) supports the mental action of coordinating the height with changes in the volume (MA1). MA1 has been identified by observing students

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that's in the bottle.

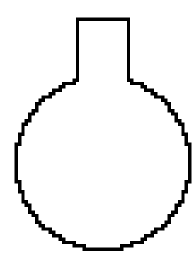

Figure 1. The Bottle Problem.

label the axes and by hearing them express an awareness that as one variable changes, the other variable changes (e.g., as volume changes, height changes). These students do not necessarily attend to the direction, amount, or rate of change.

The *Direction Level* (L2) supports both MA1 and the mental action of coordinating the direction (increasing) of change of the height while considering changes in the volume (MA2). MA2 has been identified by observing students construct an increasing straight line, or by verbalizing that as more water is added, the height of the water in the bottle increases.

The *Quantitative Coordination Level* (L3) supports MA1, MA2, and the mental action of coordinating the amount of change of the height with the amount of change of the volume while imagining changes in the volume (MA3). MA3 has been identified by observing students place marks on the side of the bottle (with each increment successively smaller until reaching the middle and successively larger from the middle to the neck). MA3 has also been identified by observing students plot points on the graph or by hearing remarks that express their awareness of how the height changes while they consider increases in the amount of water.

The *Average Rate Level* (L4) supports MA1, MA2, MA3, and the mental action of coordinating the average rate of change of the height with respect to the volume for equal amounts of the volume (MA4). MA4 has been identified in students by observing their construction of contiguous line segments on the graph, with the slope of each segment adjusted to reflect the (relative) rate for the specified amount of water; or by hearing remarks that express their awareness of the rate of change of the height with respect to the volume while they consider equal amounts of water. (Note that some students have been observed initially constructing line segments that were not contiguous and that some students have been observed switching the roles of the independent (volume) and dependent (height) variable several times in the context of discussing the thinking that they used to construct the graph for this task.)

Instantaneous Rate Level (L5) supports MA1 through MA4 and the mental action of coordinating the instantaneous rate of change of the height (with the respect to volume) with changes in the volume (MA5). MA5 has been identified in students by observing the construction of a smooth curve that is concave down, then concave up, then linear; and by hearing remarks that suggest an understanding that the smooth curve resulted from considering the changing nature of the rate while imagining the water changing continuously. It is noteworthy that a student would receive an Instantaneous Rate Level classification only if he or she demonstrated an understanding that the instantaneous rate resulted from considering smaller and smaller amounts of water (built on the reasoning exhibited in MA4). The image that supports L5 reasoning would also support behaviors that demonstrate an understanding of *why* an inflection point conveys the exact point where the rate of change of the height (with respect to volume) changed from decreasing to increasing, or from increasing to decreasing.

Some students have been observed exhibiting behaviors that gave the appearance of engaging in MA5 (e.g., construction of a smooth curve). When asked to provide a rationale for their construction, however, they indicated that they had relied on memorized facts to guide their construction. Their behavior was classified as *pseudo-analytical*, and the mental action that supported this behavior was classified as *pseudo-analytic* MA5 (Vinner, 1997).

METHOD

Participants

Twenty students who had recently completed 2nd-semester calculus with a course grade of A were asked to respond to five items that involved an analysis of covariant aspects of dynamic events (e.g., water filling a spherical shaped bottle, temperature changing over time, a ladder sliding down a wall). These students represented most of the A students from five different sections, with a different teacher for each section. The course materials used in teaching the sections were traditional, and lecture was the primary mode of instruction. Calculators were not allowed on either homework or exams. The 20 students received payment for their time spent in completing the five-item quantitative assessment. Six of these students were subsequently invited to participate in 90-minute clinical interview for which they were also paid. The selection of the interview subjects was based on assembling a collection of individuals who had provided diverse responses on the written instrument.

Procedures

The five-item instrument was completed by each of the 20 subjects within a week of completing the final exam. It was administered in a monitored setting with no time restriction, and the students were asked to provide their answers in writing. Written items were then scored using carefully developed and tested rubrics (Carlson, 1998), and the percentage of the students who provided each response type for each item was determined.

The six interviews were conducted within 2 days of the students' completion of the five-item written instrument. Although the interviews were primarily unstructured, with the interviewer spontaneously reacting to the student's description of her or his solution, prepared interview questions imposed some structure. During the interview, the researcher initially read each question aloud and made general reference to the student's written response. The student was then given a few minutes to review her or his written response and was subsequently prompted to describe and justify the solution verbally. After the student summarized the written response, the researcher made general inquiries, using prompts such as "explain" or "clarify," and continued to ask more specific questions until the student responded or appeared to have communicated all relevant knowledge. This process was repeated for each item.

Analysis of the interview results involved an initial reading of each interview transcript to determine the general nature of the response. This first reading was followed by numerous careful readings by two of the authors to classify the behaviors and responses of each student on each item, using the mental actions described in the covariation framework. After labeling the mental actions (e.g., MA1, MA2, MA3) associated with the various behaviors exhibited for a single item, both authors reviewed the entire response for that item to determine the covariational reasoning level (e.g., L3) that supported the identified mental actions that the response displayed. Inconsistencies in the coding by the two authors were resolved by discussion, with the final labeling representing agreement between the two coders. Illustrations of select quantitative data and coded interview excerpts are followed by a discussion of the students' responses to three of the five covariational reasoning tasks.

RESULTS

The Bottle Problem

The bottle problem (see Figure 1) prompted students to construct a graph of a dynamic situation with a continuously changing rate and with an instance of the rate changing from decreasing to increasing (i.e., an inflection point). Table 3 shows the kinds of responses that the 20 students provided on the written assessment. Only 5 (25%) of these high-performing 2nd-semester calculus students provided an acceptable solution, while 14 of the 20 students (70%) constructed an increasing graph that was strictly concave up or concave down.

When prompted during the follow-up interview to describe the graph's shape, the six interview subjects provided varied responses. These responses are described more fully in the next section, but we summarize some important points here. Only two of the interview subjects—Student A and Student C—provided a response that suggested an image of a continuously changing instantaneous rate (MA5) for this situation. When prompted to explain the rationale for her acceptable graph, Student A initially stated, "If you look at it as putting the same amount of water in each time and look at how much the height would change, the height would be changing more quickly, and in the middle if you add the same amount of water, the height would not change as much as it would at the bottom." (MA3). When prompted to explain why she had constructed a smooth curve, Student A responded, "I imag-

Table 3
Bottle Problem Quantitative Results

Response types	Number of students out of 20 providing each response type
Constructed a line segment with positive slope	1
Constructed an increasing concave-up graph	11
Constructed an increasing concave-down graph	3
Acceptable graph, except for slope of segment	3
All aspects of graph were acceptable	2

ined the height changing as the water was pouring in at a steady rate" (MA3). She also characterized the inflection point as the point "where the rate at which it was filling goes from decreasing to increasing" (suggestive of MA5).

By contrast, Student C, who also constructed an acceptable graph by sketching a smooth curve over her previously constructed contiguous line segments, justified the construction by saying, "I just know that it must be smooth because this is what these graphs always look like, not these connected line segments." Even though her initial construction of a smooth curve was suggestive of MA5 and appeared to depict an image of continuously changing rate, further probing revealed that this student's answer expressed only an opinion of how the graph *should* look, rather than an emergent representation of how the variables changed. This response was therefore classified as pseudo-analytic MA5.

In analyzing the thinking of the three students who provided either a concave-up or concave-down construction, we noted that two of these students (Students B and E) at times during the interview constructed images of the height changing at a varying rate (e.g., "As you go up a little more height increases and the volume increases quite a bit" [MA3]). However, inconsistencies in their reasoning appeared to result in their constructing an incorrect graph. Student B justified his concave-down construction by saying, "Every time you have to put more and more volume in to get a greater height towards the middle of the bottle" (MA3). (Notice that this illustrates a situation where the student switched the roles of the independent and dependent variables—that is, he considered the amount of change of the volume while considering uniform changes in the height). He subsequently failed to continue thinking about the relative changes in the volume and height for the top half of the bottle. Student F justified his concave-up construction by saying, "As I add more water, it still gets higher and higher" (suggestive of MA2). Although both of these students appeared to possess initial images of height changing as more water was added, at some point during the interview they appeared to focus on incorrect information or had difficulty representing their correct reasoning patterns using a graph. The remaining interview subject, Student D, provided an increasing straight line and stated with confidence, "As the volume comes up, the height would go up at a steady rate … it would be a straight line" (MA2). He appeared to notice only that the height increased while considering increases in the volume (MA2).

Both the quantitative and qualitative data for the Bottle Problem support the finding that very few of these high-performing 2nd-semester calculus students were able to form accurate images of the continuously changing instantaneous rate (MA5) for this dynamic function event. The excerpts from the student interviews that follow illustrate this finding.

Student A constructed an acceptable graph, which appears in Figure 2. During the course of the interview, she initially focused on the amount of change of the height while considering fixed increases in the volume (MA3). These comments were followed by discussions of the slope and rate changes for fixed amounts of water (MA4). When prompted for more explanation, she eventually moved to comments that conveyed that she was attending to the continuously changing rate while imagining the bottle filling with water (MA5). She appeared to understand the information conveyed by the inflection point and appeared to have a mature image of changing rate over the domain of the function. The behaviors exhibited by this student when responding to this task suggested an Instantaneous Rate (L5) covariational reasoning ability.

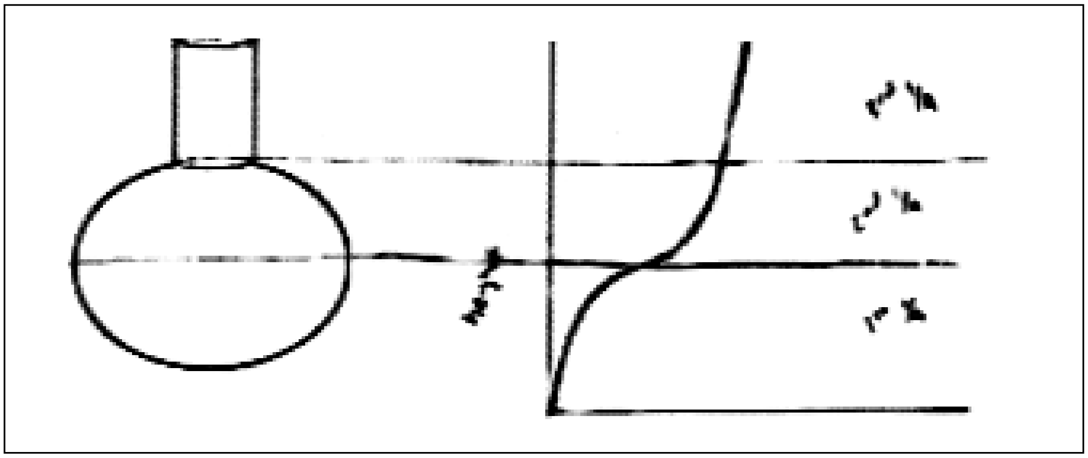

Figure 2. Student A's written response.

Int.: Describe how you sketched the graph. [*Note: Final graph is acceptable*.]

A: I knew it changed different for the bottom part because it's circular and the top part has straight walls. If you look at it as putting the same amount of water in each time and look at how much the height would change, that's basically what I was trying to do. So for the first part, the height would be changing more quickly, and in the middle if you add the same amount of water, the height would not change as much as it would at the bottom [MA3]. It's symmetric.

Int.: How does that affect the graph?

A: Higher slope in the beginning, then it levels out, then a higher slope again [MA4]. Then for the neck part it's basically a straight line because you're filling the same area with each amount [MA3].

Int.: Can you tell what happened at this point [*pointing to the inflection point*]?

A: That is where the point of symmetry is. I guess it would also be where the second derivative is equal to 0, which is where the rate at which it was filling goes from decreasing to increasing [MA5].

Int.: Why did you draw a smooth curve through the lines?

A: Well, I imagined the slope changing as the water was pouring in at a steady rate [MA5].

Int.: Do you have anything else to add? What is the slope of the straight line?

A: About like the curve right here [*pointing to the junction of the curve and line*].

Student B constructed a concave-down graph for the entire domain of the function. During the interview, he initially focused on the direction of the change of the height, as revealed by his comment, "the more water, the higher the height" (MA2). Further prompting revealed that he was conceptually able to coordinate changes in the height with changes in the amount of water (i.e., "As you go up, a little more height increases and the volume increases quite a bit" [MA3]). The statement, "You have to put more and more volume in to get a greater height toward the middle" is also indicative of MA3. His expressed behaviors were suggestive of a Quantitative Coordination (L3) covariational reasoning ability for this task. His concave-down construction appeared to result from his failure to continue coordinating changes in height with changes in the amount of water.

Int.: Explain your solution [*Note: Student has provided a concave-down graph for the entire domain.*]

B: This is my least favorite problem. I tried to solve for height in terms of volume and it was a mess.

Int.: Can you analyze the situation without explicitly solving for *h*?

B: OK, the more water, the higher the height would be [MA2]. In terms of height of the water, that is what we are talking about. If you are talking about the height left over, that is basically decreasing. Right here the height will be zero and the volume is zero. As you go up, a little more height increases and the volume increases quite a bit [MA3], so the amount by which the height goes up is not as fast [MA3]. Once you get there [*pointing to halfway up the spherical part of the bottle*], the height increases even slower [MA3]. I guess from here to there height increases the same as the volume increases, and once you get here it increases slower [MA3]. No, I am wrong. So, every time you have to put more and more volume in to get a greater height towards the middle of the bottle and once you get here, it would be linear, probably [*pointing to the top of the spherical portion*]. So, it's always going up [*tracing his finger along the concave-down graph*], then it would be a line.

Int.: So, what does the graph look like?

B: Like this [*pointing to the concave-down graph he has constructed*], but it has a straight line at the end.

Student C produced a graph with only minor errors. Her initial justification that "it is going to be filling rapidly, so you are going to have greater slope" focused on the relative slope for a section of the graph. She immediately followed this statement by the further justification, "As you increase the volume you are going to get less height" (MA3), and her final justification was a statement of rules learned in calculus. Her responses suggested that although she was able to associate a greater slope with the bottle's filling rapidly and appeared at times to be imagining continuously changing instantaneous rate (MA5), she did not seem to understand how the instantaneous rates were obtained (i.e., she was not able to unpack MA5). Not even in response to direct probing could she explain what the inflection point conveyed. As a result, she was not classified as having L5 covariational reasoning ability. When responding about this task, Student C appeared to use L3 reasoning predominantly, along with rules learned and memorized in calculus. This combination of abilities appeared to be adequate for the construction of an acceptable graph.

Int.: Explain how you obtained your graph. [*Note: Final graph is acceptable.*]

C: I knew it was filling at a cubic rate somehow, so it would have something like a cubic equation. When you take the inverse of that equation it whips it like that. But I was also able to see here that when you start out, it's going to be filling rapidly, so you are going to have a greater slope [MA5—*the student appears to know that "rapidly" and "more steep" are connected but does not demonstrate an understanding of how the instantaneous rate was obtained*. *Shows some confusion and continues:*] But as you increase the volume, you're going to get less of a height change until you get up to here [MA3]. As you get past the halfway point, it's going to go from concave down to concave up and you're going to have an inflection point. For this cylinder part, I know it's going to be linear, since for the cylinder it's related by volume, which equals area times height. And so we have area as a constant. So what we have is a linear equation for height as it's related to volume.

Int.: Can you tell me why you drew the smooth curve through the line segments that you had constructed?

C: Well [*a long pause*] … I just know that it must be smooth because this is what these graphs always look like, not these connected line segments [pseudo-analytic MA5].

Int.: Can you tell me why it changed concavity there [*pointing to the inflection point*]?

C: Because if you take the second derivative of this volume in terms of height, you'll get a 0. On this side you have a negative acceleration. But once you reach the halfway point, then you start becoming a positive second derivative.

Student D constructed an increasing straight line for his solution. During the interview, he appeared to coordinate only the direction of the change in the height while considering changes in the volume (MA2). The behaviors exhibited by this student when responding to this task were suggestive of a Direction (L2) covariational reasoning ability.

Int.: Can you explain your solution? [*Note: Student's solution is an increasing straight line.*]

D: I tried to solve for *h*. But I think I need to define it as a piecewise defined function. Maybe then I can figure it out.

Int.: Did you try to get an idea of the general shape of the graph by imagining the bottle filling with water?

D: As the volume comes up, the height would go up at a steady rate [MA1, MA2].

Int.: How would you represent this graphically?

D: It would be a straight line [*passes his hand over the increasing straight line*].

Int.: So, the entire graph is a straight line.

D: Yes.

Student E provided a concave-up graph for his written solution. When probed to explain his answer, he replied that "as you added more water, the height was going up" (MA2). He then proceeded to explain his concave-up graph with the following statement, "The amount by which the height goes up is increasing" (MA3). However, his factual information was flawed (the amount of the height change was decreasing). Because he did not consistently exhibit behaviors supported by MA3, he was classified as having a Direction (L2) covariational reasoning ability.

Int.: Can you describe how you determined your graph? [*Note: Student has provided a concave-up graph*.]

E: Well, I knew that as you added more water the height was going to go up [MA2] … um.… Then I knew that it would curve up because your graph is getting higher all the time since the height is always increasing [MA3]. So it is concave up [*points to the concave-up graph*].

Int.: What is increasing?

E: The amount by which the height goes up is increasing [MA3]. This means that it will curve up like this.

Int.: How do you explain what the shape looks like here [*pointing to the middle of the bottle*]?

E: It is still true here that as you add more water, it will increase in height, so it curves up here too [MA2].

Student F also constructed a concave-up graph and appeared to focus consistently on the amount of change of the height while considering changes in the volume (MA3), as revealed by his justification, "As I add more water it still gets higher and higher." At one point during the interview, he indicated that "height is going up more and more" [MA3]. However, he did not persist long enough to resolve the incorrectness of this statement (when imaging water being added to the lower half of the bottle); nor did he follow through in resolving the inconsistency that he noticed later in the interview (see the following excerpt). He did not show a consistent pattern of behaviors supported by MA3. Consequently, the behaviors exhibited by this student when responding to this task were suggestive of Direction (L2) covariational reasoning ability.

Int.: Can you explain how you determined your graph [*Note: Student has provided a concave-up graph*.]

F: When you're given a flask like this, the way I thought of it was, you have to start the coordinates at (0, 0) with volume equal to 0, and the height equal to 0. When you start filling something that has such a wide base like this, the height is going to increase as fast as the volume [MA1, MA2]. Then as more water is added it gets higher and higher, so the graph goes up more and more [MA3; *pointing to the concave-up graph*].

Int.: What happens at the middle of the spherical portion?

F: Now, I am confused. Will it continue to go up higher and higher? [*Pauses*.] Well yes, as I look at the part above the middle, as I add more water it still gets higher and higher so yes, it curves up like this [MA3; *again pointing to the concave-up graph*].

The Temperature Problem

The task shown in Figure 3 presented students with a rate-of-change graph and then called on them to construct the corresponding temperature graph. This problem required students to directly interpret rate information displayed in the form of a graph and to use the information to graph the original function based on temperature. As shown in Table 4, four (20%) of the high-performing 2nd-semester calculus students constructed an acceptable temperature graph, given the rate of change of temperature for an 8-hour period; five (25%) of the students produced the same graph for the temperature graph as the one given for the rate-of-change graph. We also found that six (30%) failed to note the concavity changes when constructing their graphs. The six students who omitted the concavity changes provided a concave-down graph from $t = 0$ to $t = 6$, with a maximum value at $t = 2$.

Given the graph of the rate of change of the temperature over an 8-hour time period, construct a rough sketch of the graph of the temperature over the 8-hour time period. Assume the temperature at time $t = 0$ is 0 degrees Celsius.

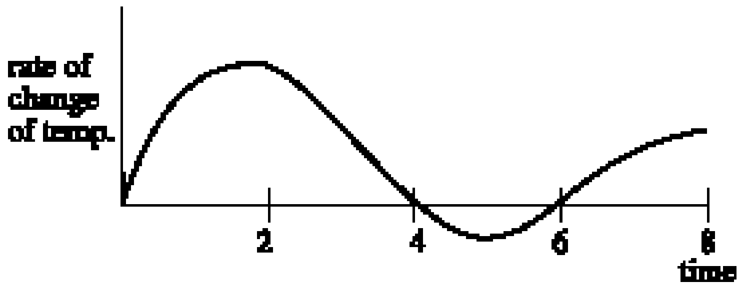

Figure 3. The Temperature Problem.

Table 4
Temperature Problem Quantitative Results

Response type	Number of students out of 20 providing each response type
Constructed a strictly concave-up graph for the entire domain	1
Constructed the same graph as the temperature graph	5
Omitted the concavity changes at $t = 2$ and $t = 5$	6
Reversed the concavity	4
All aspects of graph were acceptable	4

The follow-up interviews revealed that, of those four students who provided an acceptable response, like Student C's response shown in Figure 4, there was little evidence that they were interpreting the rate information conveyed by the graph. When Student C was prompted to justify her acceptable response, she replied, "Positive first derivative implies function increasing, negative first derivative implies function decreasing," and "Second derivative equal to zero occurs at inflection points." When asked to explain the reasoning that led to these statements, she indicated that this was how she had learned the information in class, and she didn't know how to think about it any other way (pseudo-analytic MA5). It is interesting to note that even when her responses were directly probed, she also appeared unable to construct an image of the temperature changing while imagining changes in time (MA3). Her response suggested that a memorized set of rules guided her construction. However, this is not surprising if one considers the nature of a traditional calculus course.

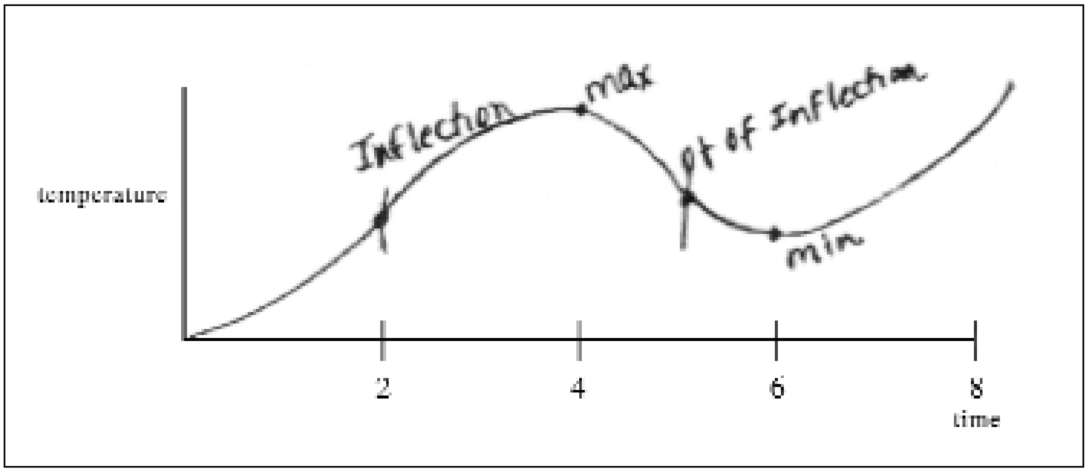

Figure 4. Student C's response to the Temperature Problem.

Moreover, the collection of follow-up interviews revealed that most of these students did not construct an accurate image of the rate changing (MA4) as they considered increases in the domain. The two students who constructed an increasing temperature graph (from $t = 0$ to $t = 4$) did not appear to understand what was being conveyed when the rate began to decrease at $t = 2$. When prompted to explain, both students indicated that because the rate graph was positive from $t = 0$ to $t = 4$, the temperate graph must be increasing. When specifically asked to explain the behavior of the temperature graph at $t = 2$, one of these two students commented that "*y* at 2 it is also positive, so it will keep curving up until it is 4." Even though these students appeared to have an initial image of the temperature function increasing at an increasing rate (MA5), their inability to note and represent the rate changing from increasing to decreasing (i.e., the inflection point), as shown by their concave-up construction and remarks, suggested weaknesses in their understandings.

Another student who had constructed the same graph for the temperature graph as the given rate-of-change graph, said, "This is hard to think about…. It is hard for me not to just draw the shape that I see … it really throws me off." This student appeared to make no attempt to interpret the rate-of-change information displayed by the graph. Rather, he appeared to want to reconstruct the same graph that he was observing.

The Ladder Problem

The ladder problem shown in Figure 5 is a modification of a problem reported in Monk (1992), which prompted students to select a means of representing a dynamic situation (i.e., a ladder moving down a wall). The results of student responses to this task appear in Table 5. When prompted to describe the speed of the top of a ladder as the bottom of the ladder is pulled away from a wall, eight (40%) of these 2nd-semester calculus students provided an accurate justification for the claim that the top of the ladder would speed up as the bottom of the ladder is pulled away from the wall. Five additional students (25%) also indicated that the top of the ladder would speed up, but provided no justification to support the claim. In addition, five students conveyed the idea that the speed of the top of the ladder would be constant, and two students (10%) indicated it would slow down.

> From a vertical position against a wall, a ladder is pulled away at the bottom at a constant rate. Describe the speed of the top of the ladder as it slides down the wall. Justify your claim.

Figure 5. The Ladder Problem.

Table 5
Ladder Problem Quantitative Results

Response type	Number of students out of 20 providing each response type
Speeds up—valid written justification	8
Speeds up—no justification	5
Stays the same	5
Slows down	2

The justifications provided on the written instrument revealed that the eight students who had provided a correct response with a valid justification had imagined a physical enactment of the ladder falling down the wall. This observation was based on a succession of pictures of the ladder in different positions drawn by the students and/or their written explanations. The follow-up interviews with two of these students supported this observation.

When one of these students (Student B) was prompted to explain his correct response, he performed a physical enactment of the situation, using a pencil and book on a table. As he successively pulled the bottom of the pencil away from the book by uniform amounts, he explained, "As I pull the bottom out, the amount by which the top drops gets bigger as it gets closer to the table" (MA3). His comments suggested that he was observing the varying amounts by which the top of the pencil dropped toward the table as the bottom was pulled out by uniform amounts. His explanation appeared to involve the coordination of an image of the magnitude of the change in the dependent variable with uniform changes in the independent variable (MA3). Student A provided a similar response, except that her enactment involved using her hand and a book to model the situation. She began by pressing her flat hand against the book and successively moved the bottom of her hand away while watching the amount by which the top of her hand dropped down. Both students appeared to assume that the greater fall implied speeding up; however, specific prompts were not offered to verify this assumption, nor were specific prompts used to elicit the reasoning behind this deduction.

Two of the interview subjects provided no justification on the written instrument to explain their correct responses. However, when prompted during the interview to explain their reasoning, Student E did provide a valid justification that also used a self-constructed enactment of the situation. Student D indicated that he had "just guessed." It is not clear whether he constructed an image of the situation as a basis for his guess.

The remaining two interview subjects indicated that the speed of the top of the ladder would remain constant as the bottom was pulled away from the wall. These students both drew pictures of the ladder in different positions, but modified the length of the ladder so that the amounts of the drop remained the same for each new position of the ladder. When asked to explain their reasoning, both students provided responses that indicated they had attempted to enact the situation, but their model was flawed. Student C drew a picture of the successive positions of the ladder as the bottom was pulled out by equal amounts. The drawing also illustrated equal drops of the top of the ladder, a configuration that the student achieved by violating a condition of the problem and adjusting the length of the ladder. Even though her answer was incorrect, she appeared to engage in a behavior that suggested that she was attempting to coordinate the amount of change in the dependent variable with the change in the independent variable (MA3).

The use of physical enactment appeared to provide a powerful representational tool that assisted these students in reasoning about the change in one variable while concurrently attending to the change in the other variable. Further exploration of this observation is needed.

CONCLUSIONS

The students in this study varied in their ability to apply covariational reasoning when analyzing dynamic events. Observed trends suggest that this collection of calculus students had difficulty constructing images of a continuously changing rate, with particular difficulties in representing and interpreting images of increasing rate and decreasing rate for a physical situation (MA5). Despite these difficulties, most of the students were able to determine the general direction of the change in the dependent variable with respect to the independent variable (L2) and were frequently able to coordinate images of the amount of change of the output variable while considering changes in the input variable (MA3). However, we observed weaknesses in their ability to interpret and represent rate-of-change information (MA4; see Tables 3 and 4). Aided by the use of kinesthetic enactment, however, these students were more often able to observe patterns in the changing magnitude of the output variable (MA3), as well as patterns in the changing nature of the instantaneous rate (MA5). Nonetheless, their difficulty in viewing an instantaneous rate by imagining smaller and smaller refinements of the average rate of change appeared to persist. More importantly, this limitation (an inability to unpack MA5) appeared to create difficulties for them in accurately interpreting and understanding the meaning of an inflection point and in explaining why a curve was smooth. Even direct probing of the few students that were able to engage in MA5 revealed that they were not able to explain how the instantaneous rate was obtained. This weakness appeared to result in difficulties for them in bringing meaning to their constructions and graphical interpretations.

Despite the fact that the subjects of our study were high-performing 2nd-semester calculus students who had successfully completed a course emphasizing rate and changing rate, the majority did not exhibit behaviors suggestive of MA5 while analyzing and representing dynamic function events. They appeared to have difficulty characterizing the nature of change while imagining the independent variable changing continuously. In summary, the majority of these calculus students—

- *were able* to apply L3 reasoning consistently. They exhibited behaviors that suggested they were able to coordinate changes in the direction and amount of change of the dependent variable in tandem with an imagined change of the independent variable (MA1, MA2 and MA3);
- *were unable* to apply L4 reasoning consistently. They exhibited behaviors that suggested they were unable to consistently coordinate changes in the average rate of change with fixed changes in the independent variable for a function's domain (MA1 to MA4);
- *had difficulty* applying L5 reasoning. They were not consistently able to exhibit behaviors that suggested that they were able to coordinate the instantaneous rate of change with continuous changes in the independent variable (MA5);
- *had difficulty* explaining why a curve is smooth and what is conveyed by an inflection point on a graph (i.e., applying L5 covariational reasoning).

Our results support the work of Confrey and Smith (1995) and Thompson (1994a), who revealed similar findings regarding the complexity of reasoning about covarying relationships; however, our study extends what has previously been reported by identifying specific aspects of covariational reasoning that appear to be problematic for college-level students. It is also our hope that the study's results and the covariation framework will serve to explicate the cognitive actions involved in students' reasoning in interpreting and representing dynamic function events.

DISCUSSION

Research has revealed that the basic idea of covariation is accessible to elementary and middle school children (Confrey & Smith, 1994; Thompson, 1994c). It seems reasonable to think, then, that this same idea would also be accessible to high-performing 2nd-semester calculus students. Therefore, the results of this study raise concerns, especially when we consider that the selected tasks could be completed successfully by students with no knowledge of calculus but with a strong covariational reasoning ability (Carlson & Larsen, in press). Since the information assessed in this study was intended to be foundational for building and connecting the major ideas of calculus, we believe these findings suggest a need to monitor the development of students' understandings of function and their covariational reasoning abilities prior to and during their study of calculus. As this study and others have revealed, even high-performing students can emerge from 2nd-semester calculus with superficial understandings of ideas that are foundational for future study of mathematics and science. Our failure to monitor these understandings and reasoning abilities portends negative consequences for students.

The thinking revealed in this study should prove useful for informing the design and development of curricular materials aimed at promoting students' covariational reasoning abilities. The results also underscore the need for students to have opportunities to think about the covariational nature of functions in real-life dynamic events. We recommend that students be given lines of inquiry that compel probing reflections on their own understandings of patterns of change (involving changing rates of change). Accordingly, we believe that curricula at the high school and university levels should take into consideration the complexity of acquiring L5 (Instantaneous Rate) reasoning and should provide curricular experiences that sustain and promote this reasoning ability, especially when one considers its importance for understanding major concepts of calculus (e.g., limit, derivative, accumulation) and for representing and understanding models of dynamic function events.

The theoretical model and results from this study should also be useful for classroom teachers in both identifying and promoting the development of their students' covariational reasoning abilities. Curricular activities that support the covariational approach to instruction are under development by the authors and have been administered to both preservice secondary teachers in a methods course and 1st-semester calculus students at a large public university in the southwestern United States. The development of these curricular activities was guided by the covariation framework and the insights gained from this study. Preliminary observations of students working with this curriculum have revealed positive shifts in their covariational reasoning abilities. Although the curriculum will benefit from multiple refinements as students' responses continue to suggest ideas for improvements, these observations are encouraging.

The new century offers educators a plethora of technologies, including graphing calculators, geometry software, computer algebra systems, electronic laboratory probes, specialty software such as MathCars (Kaput, 1994), and specially designed physical devices (e.g., Monk & Nemirovsky, 1994) for studying real-time dynamic events. Rich pedagogical opportunities abound for building on students' intuition about and experience with dynamically changing quantities. Properly grounded and coupled with sufficient teacher training, these technologies offer valuable tools for students in learning to apply covariational reasoning to analyze and interpret dynamic function situations.

FUTURE RESEARCH

The work from this investigation has resulted in our reflecting on the nature of the reasoning patterns involved in applying covariational reasoning. We claim that conceptually coordinating the change in one variable with changes in the other variable while attending to how the variables change in relation to each other involves a mental enactment of the operation of coordinating on two objects (these objects are different depending on the mental action in the framework). This observation led us to hypothesize that the mental actions involved in applying covariational reasoning are characteristic of *transformational reasoning* as described by Simon (1996):

> Transformational reasoning is the mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or continuum of states are generated. (p. 201)

We view the mental actions that we have described in the Covariation Framework as examples of transformation reproductive images (i.e., the problem solver is able to visualize the transformation resulting from an operator). In our case, the student visualizes the transformation of a dynamic situation as resulting from the operation of coordinating. When engaging in MA3, the student is able to visualize the transformation of a dynamic situation (e.g., a ladder falling down the wall, a bottle filling with water) by performing a mental enactment of coordinating two objects (the amount of change in one variable with an amount of change in another variable); while MA5 involves a mental enactment of coordinating the instantaneous rate of change in one variable with changes in the other variable. In both cases, the mental enactment on the objects results in a transformation of the system (e.g., the ladder is envisioned as being in a different position, the bottle is envisioned as containing more water).

Although we claim that we have observed instances of transformational reasoning, we offer no information about the process of coming to generate a particular transformational approach. We concur with Simon (1996) in calling for explorations of this question, as our results also support the notion that appropriate application of transformational reasoning may prove to be extremely powerful for understanding and validating a mathematical system.

Our investigation also calls for an extension of the Covariation Framework to include a greater level of epistemological refinement for understanding covarying quantities. Such a framework may include aspects of concept development as it relates to covariational reasoning abilities. It may also include a more finely grained analysis of L5 (Instantaneous Rate) reasoning. In addition, it could be extended to articulate the nature of covariational reasoning more clearly in the context of working with formulas or the algebraic form of a function.

Our research invites future study of some specific questions on the centrality of continuity and the implicit time variable in covariational reasoning. Our discussion and instruments deal with physical relationships that are inherently continuous. It is unclear to what extent the framework applies to students' study of discontinuous dynamic function events. Furthermore, our experience suggests that students have a powerful tendency to think of time as a variable, even to the point of introducing it into situations like that in the Bottle Problem, where it is not strictly necessary (or requested). Future research may clarify the role of the implicit time variable in the development of student's covariational reasoning.

Another promising area of research includes investigations of the effectiveness of various curricular interventions in developing students' ability to apply covariational reasoning when solving problems that involve real-world dynamic situations. Such studies may also provide information about the effect of taking a covariational approach to learning functions on students' development of their understanding of the function concept in general.

We have provided examples of students who appeared to be able to apply covariational reasoning to the bottle problem and the ladder problem in a kinesthetic context but who were unable to use the same reasoning patterns when attempting to construct a graph (i.e., to reason in the graphing representational context) for these situations. These examples are important because they suggest that the covariation framework may be used to infer information not just about the developmental level of student images of covariation but also about the internal structure of these images. If we imagine that a student's overall image of covariation of a dynamic situation contains specific images related to each relevant representation system, we may be able to use the framework to analyze the way in which these images are connected and coordinated.

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